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SUMMARY

The steady-state incompressible Navier-Stokes equations in two dimensions are solved numerically using the artificial compressibility formulation. The convective terms are upwind-differenced using a flux-difference split approach that has uniformly high accuracy throughout the interior grid points. The viscous fluxes are differenced using second-order-accurate central differences. The numerical system of equations is solved using an implicit line-relaxation scheme. Although the current study is limited to steady-state problems, it is shown that this entire formulation can be used for solving unsteady problems. Characteristic boundary conditions are formulated and used in the solution procedure. The overall scheme is capable of being run at extremely large pseudotime steps, leading to fast convergence. Three test cases are presented to demonstrate the accuracy and robustness of the code. These are the flow in a square-driven cavity, flow over a backward-facing step, and flow around a two-dimensional circular cylinder.

INTRODUCTION

The motivation for the current work is a desire to find an efficient method of solution for the incompressible Navier-Stokes equations for complex three-dimensional (3-D) geometries. Several methods exist which reformulate the equations into a nonprimitive-variable form. These include a velocity-vorticity method as proposed by Fasel [1], and a vector potential-vorticity method proposed by Aziz and Hellums [2]. Each of these methods requires the solution of three Poisson equations at each time level, which is one reason why these methods have not become popular. Further work on the velocity-vorticity method and the use of a direct solver is being done by Hafez, Dacles, and Soliman [3]. These methods currently appear to be too expensive for solving large 3-D problems. This leaves methods formulated in primitive variables. The primary difficulty in solving the incompressible Navier-Stokes equations in primitive variables stems from the lack of a time derivative in the continuity equation. There is no straightforward way to iteratively march these equations in time and ensure a divergence-free velocity field. The marker-and-cell (MAC) method of Harlow and Welch [4] accomplishes this by solving for the velocity field from the momentum equations and for the pressure field from a Poisson equation which is derived by taking the divergence of the momentum equations. This method can be very costly because of the need to solve the Poisson equation at each time iteration, and because there is no strong coupling between the velocity field and the pressure field.

Another approach which has been used extensively is that of the fractional-step method originally introduced by Chorin [5], and used by Kim and Moin [6], and Rosenfeld, Kwak, and Vinokur [7]. This method advances the solution in time using two (or more) steps. First the momentum equations are solved for an intermediate velocity field which will generally not be divergence-free. In the next step, a pressure field is obtained which will map the intermediate velocity field into a

divergence-free field, thus obtaining the solution at the next time level. The second step generally requires the solution of a Poisson equation in pressure.

The method chosen for the current work is known as the pseudocompressibility method and was first introduced by Chorin [8]. It has been used with much success by Kwak *et al.* [9] for solving complex incompressible flow problems in generalized coordinates. In this formulation, a time derivative of pressure is added to the continuity equation. Together with the momentum equations, this forms a hyperbolic system of equations which can be marched in pseudotime to a steady-state solution. The method can also be extended to solve time-dependent problems [10,11] by using subiterations in pseudotime at each physical time step. If all that is desired is the steady-state solution to a problem, the pseudocompressibility method can be a very efficient formulation because it does not require that a divergence-free velocity field be obtained at each iteration. The addition of the time derivative of pressure to the continuity creates a hyperbolic system of equations complete with artificial pressure waves of finite speed. When the solution converges to a steady state, a divergence-free solution is obtained. Since this is the case, many of the well-developed compressible flow algorithms can be used for this method.

Many previous applications of this method have used central differencing of the convective fluxes. This approach also requires that artificial dissipation be explicitly added in order to damp out the spurious oscillations which are a result of the nonlinearity of the convective fluxes. Such a scheme can be difficult to apply because the artificial dissipation parameters must be adjusted for each specific calculation. The use of the artificial dissipation will also tend to hamper the accuracy of the calculations [12]. To avoid the problems associated with central differencing, an upwind-differencing scheme is considered here. Of most recent interest has been the class of upwind-differencing schemes which bias the differencing based on the sign of the eigenvalues of the convective flux Jacobians. A number of these types of schemes have been developed in conjunction with solving the Euler equations and the compressible Navier-Stokes equations [13-16]. The development of these schemes base the upwind differencing on the physics of the Riemann problem. In the case of the pseudocompressibility method, the upwind differencing is merely a way of using the physics of the artificial waves to obtain a smooth numerical solution.

Much of the current development of upwind-differencing schemes has focused on the ability to resolve sharp discontinuities without kinks or overshoots. By limiting the order of the differencing at points near the discontinuities, and thereby increasing the dissipation provided by the differencing, the schemes have the total variation diminishing (TVD) property. Applications of TVD schemes to the incompressible equations were done by Hartwich and Hsu [17,18], and Gorski [19]. These investigators were able to obtain 3-D solutions which were third-order-accurate in the convective terms, except near regions of large change in gradient, where the order of the differencing was reduced to increase the amount of dissipation added.

Since solutions to the incompressible equations do not have strong discontinuities such as shocks, it is reasoned that the incompressible equations could be solved without the need for any limiting, and that flux-difference splitting of uniformly high order could be used. This paper attempts to show that this is so by using a flux-difference splitting type of formulation similar to that used for compressible flow in [13,14]. The current work concentrates on developing this scheme with the use of a two-dimensional (2-D) flow solver using fifth-order upwind differencing of the convective terms. Since the development of the upwind-differencing schemes considered here is based upon an analysis of a one-dimensional (1-D) hyperbolic conservation law, the use of a 2-D code for the initial testing done here will not be out of line from the desired goal of a 3-D algorithm. This will expedite much of the code development because of the smaller computational requirements of a 2-D code and because of the relative ease with which 2-D results can be analyzed, compared, and presented.

In the following sections, the details of the 2-D code are presented, including the governing equations and the similarity transformation for the Jacobian matrix of the convective fluxes. The specific details of the upwind scheme are given, followed by the details of the implicit line-relaxation scheme used to solve the equations. Some boundary conditions based on the method of characteristics have been developed, and are presented. The computed results section shows the robustness and accuracy of the code by presenting three sample problems—the flow inside a driven cavity, the flow over a backward-facing step, and the flow over a circular cylinder.

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GOVERNING EQUATIONS

The equations governing constant-density viscous flow are presented here in nondimensional form. Following the pseudocompressibility formulation, a time derivative of pressure is added to the continuity equation resulting in

$$\frac{\partial p}{\partial \tau} + \beta \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \quad (1)$$

where u and v are velocity components in the x and y directions, respectively, and β is known as the pseudocompressibility constant. Here, τ represents pseudotime and is not related in any way to physical time. Combining equation (1) with the momentum equations for the incompressible Navier-Stokes equations results in the following system in Cartesian coordinates

$$\begin{aligned} \frac{\partial}{\partial \tau} D + \frac{\partial}{\partial x} (E - E_v) + \frac{\partial}{\partial y} (F - F_v) &= 0 \\ D &= \begin{bmatrix} p \\ u \\ v \end{bmatrix} \quad E = \begin{bmatrix} \beta u \\ u^2 + p \\ uv \end{bmatrix} \quad F = \begin{bmatrix} \beta v \\ uv \\ v^2 + p \end{bmatrix} \\ E_v &= \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \end{bmatrix} \quad F_v = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \end{bmatrix} \\ \tau_{xx} &= 2\nu \frac{\partial u}{\partial x} \quad \tau_{yy} = 2\nu \frac{\partial v}{\partial y} \\ \tau_{xy} &= \tau_{yx} = \nu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \end{aligned} \quad (2)$$

where p is the pressure. In this formulation the Reynolds stress has been approximated as a function of the strain-rate tensor, and thus ν represents a sum of the kinematic viscosity and the turbulent eddy viscosity. The equations in (2) are transformed into generalized curvilinear coordinates given by

$$\begin{aligned} \xi &= \xi(x, y) \\ \eta &= \eta(x, y) \end{aligned}$$

The equations are then given by

$$\frac{\partial}{\partial \tau} \hat{D} + \frac{\partial}{\partial \xi} (\hat{E} - \hat{E}_v) + \frac{\partial}{\partial \eta} (\hat{F} - \hat{F}_v) = 0; \quad \hat{D} = \frac{1}{J} \begin{bmatrix} p \\ u \\ v \end{bmatrix} \quad (3)$$

where J is the Jacobian of the transformation, the metrics of the transformation are

$$\xi_x = \frac{\partial \xi}{\partial x}, \quad \eta_y = \frac{\partial \eta}{\partial y}, \quad \text{etc.}$$

and the convective fluxes are given by

$$\hat{E} = \frac{1}{J} \begin{bmatrix} \beta U \\ uU + \xi_x p \\ vU + \xi_y p \end{bmatrix} \quad \hat{F} = \frac{1}{J} \begin{bmatrix} \beta V \\ uV + \eta_x p \\ vV + \eta_y p \end{bmatrix} \quad (4)$$

where the contravariant velocity components, U and V are defined as

$$U = \xi_x u + \xi_y v \\ V = \eta_x u + \eta_y v$$

In deriving the viscous fluxes, constant viscosity was assumed for simplicity and because initially only laminar flow calculations are being performed. This simplification is not necessary and will be removed in the future. The viscous fluxes are given by

$$\hat{E}_v = \frac{\nu}{J} \begin{bmatrix} 0 \\ (\xi_x^2 + \xi_y^2) u_\xi + (\xi_x \eta_x + \xi_y \eta_y) u_\eta \\ (\xi_x^2 + \xi_y^2) v_\xi + (\xi_x \eta_x + \xi_y \eta_y) v_\eta \end{bmatrix} \\ \hat{F}_v = \frac{\nu}{J} \begin{bmatrix} 0 \\ (\xi_x \eta_x + \xi_y \eta_y) u_\xi + (\eta_x^2 + \eta_y^2) u_\eta \\ (\xi_x \eta_x + \xi_y \eta_y) v_\xi + (\eta_x^2 + \eta_y^2) v_\eta \end{bmatrix} \quad (5)$$

where ν is the kinematic viscosity. If an orthogonal grid is assumed, the viscous fluxes reduce to

$$\hat{E}_v = \frac{\nu}{J} \begin{bmatrix} 0 \\ (\xi_x^2 + \xi_y^2) u_\xi \\ (\xi_x^2 + \xi_y^2) v_\xi \end{bmatrix} \quad \hat{F}_v = \frac{\nu}{J} \begin{bmatrix} 0 \\ (\eta_x^2 + \eta_y^2) u_\eta \\ (\eta_x^2 + \eta_y^2) v_\eta \end{bmatrix} \quad (6)$$

The upwind scheme requires the use of the eigensystem of the Jacobian matrix of the convective flux vectors. The 2-D eigensystem is presented here. For the 3-D equations, see Rogers, Chang, and Kwak [20], or Hartwich and Hsu [17]. Beware, however, that the transformation given by the latter can become singular for certain values of metrics.

The generalized flux vector for the 2-D system of equations is given by

$$\hat{E}_i = \begin{bmatrix} \beta Q \\ uQ + k_x p \\ vQ + k_y p \end{bmatrix} \quad (7)$$

where $\hat{E}_i = \hat{E}, \hat{F}$ for $i = 1, 2$, respectively, and the metrics are represented by

$$k_x = \frac{1}{J} \frac{\partial \xi}{\partial x}, \quad i = 1, 2 \\ k_y = \frac{1}{J} \frac{\partial \xi}{\partial y}, \quad i = 1, 2$$

where $\xi_1 = \xi$, $\xi_2 = \eta$, and the scaled contravariant velocity component is

$$Q = k_x u + k_y v$$

The Jacobian matrix of this generalized flux vector is given by

$$\hat{A}_i = \frac{\partial \hat{E}_i}{\partial D} = \begin{bmatrix} 0 & \beta k_x & \beta k_y \\ k_x & k_x u + Q & k_y u \\ k_y & k_x v & k_y v + Q \end{bmatrix} \quad (8)$$

A similarity transform for the Jacobian matrix is derived here of the form

$$\hat{A}_i = X_i \Lambda_i X_i^{-1}$$

where

$$\Lambda_i = \begin{bmatrix} Q & 0 & 0 \\ 0 & Q + c & 0 \\ 0 & 0 & Q - c \end{bmatrix} \quad (9)$$

and where c is the scaled artificial speed of sound given by

$$c = [Q^2 + \beta(k_x^2 + k_y^2)]^{1/2} \quad (10)$$

This variable will always be greater than Q , so the second eigenvalue will always be positive and the third eigenvalue will always be negative. The matrix of the right eigenvectors is given by

$$X_i = \frac{1}{2\beta c^2} \begin{bmatrix} 0 & c\beta & -c\beta \\ -2\beta k_y & u(c+Q) + \beta k_x & u(c-Q) + \beta k_x \\ 2\beta k_x & v(c+Q) + \beta k_y & v(c-Q) + \beta k_y \end{bmatrix} \quad (11)$$

and its inverse is given by

$$X_i^{-1} = \begin{bmatrix} k_y u - k_x v & -Qv - \beta k_y & Qu + \beta k_x \\ c - Q & \beta k_x & \beta k_y \\ -c - Q & \beta k_x & \beta k_y \end{bmatrix} \quad (12)$$

UPWIND DIFFERENCING

Upwind differencing is used to numerically compute the convective-flux derivatives. The upwind-differencing scheme is derived from 1-D considerations, and then is applied to each coordinate direction separately. Flux-difference splitting is used here to bias the differencing based on the sign of the eigenvalues of the convective-flux Jacobian. The scheme as presented here was originally derived by Roe [14] as an approximate Riemann solver for the compressible gasdynamics equations.

The derivative of the convective flux in the ξ direction is approximated by

$$\frac{\partial E}{\partial \xi} \approx \frac{(\tilde{E}_{j+1/2} - \tilde{E}_{j-1/2})}{\Delta \xi} \quad (13)$$

where $\tilde{E}_{j+1/2}$ is a numerical flux and j is the discrete spatial index for the ξ direction.

The numerical flux is given by

$$\tilde{E}_{j+1/2} = \frac{1}{2} [\hat{E}(D_{j+1}) + \hat{E}(D_j) - \phi_{j+1/2}] \quad (14)$$

For $\phi_{j+1/2} = 0$ this represents a second-order, central-difference scheme. The $\phi_{j+1/2}$ is a dissipation term. A first-order upwind scheme is given by

$$\phi_{j+1/2} = \Delta E_{j+1/2}^+ - \Delta E_{j+1/2}^- \quad (15)$$

where ΔE^\pm is the flux difference across positive or negative traveling waves. The flux difference is computed as

$$\Delta E_{j+1/2}^\pm = A^\pm(\bar{D})\Delta D_{j+1/2} \quad (16)$$

where the Δ operator is given by

$$\Delta D_{j+1/2} = D_{j+1} - D_j$$

The plus (minus) Jacobian matrix has only positive (negative) eigenvalues and is computed from

$$\begin{aligned} A^\pm &= X_1 \Lambda_1^\pm X_1^{-1} \\ \Lambda_1^\pm &= \frac{1}{2}(\Lambda_1 \pm |\Lambda_1|) \end{aligned} \quad (17)$$

This Jacobian matrix is evaluated using some intermediate value which is a function of the surrounding points, j and $j + 1$. The Roe properties [14], which are necessary for a conservative scheme, are satisfied if this is taken as the average of the surrounding values. Thus

$$\bar{D} = \frac{1}{2}(D_{j+1} + D_j) \quad (18)$$

A scheme of arbitrary order may be derived using these flux differences. Implementation of higher-order-accurate approximations in an explicit scheme do not require significantly more computational time if the flux differences ΔE^\pm are all computed at once for a single line. A third-order upwind flux is defined by

$$\phi_{j+1/2} = -\frac{1}{3}(\Delta E_{j-1/2}^+ - \Delta E_{j+1/2}^+ + \Delta E_{j+1/2}^- - \Delta E_{j+3/2}^-) \quad (19)$$

The primary problem with using schemes of accuracy greater than third order occurs at the boundaries. Large stencils will require special treatment at the boundaries, and a reduction of order is necessary. Therefore, when going to a higher-order-accurate scheme, compactness is desirable. Such a scheme was derived by Rai [21] using a fifth-order-accurate, upwind-biased stencil. A fifth-order, fully upwind difference would require 11 points, but this upwind-biased scheme requires only seven points. It is given by

$$\begin{aligned} \phi_{j+1/2} = -\frac{1}{30} [& -2\Delta E_{j-3/2}^+ + 11\Delta E_{j-1/2}^+ - 6\Delta E_{j+1/2}^+ - 3\Delta E_{j+3/2}^+ \\ & + 2\Delta E_{j+5/2}^- - 11\Delta E_{j+3/2}^- + 6\Delta E_{j+1/2}^- + 3\Delta E_{j-1/2}^-] \end{aligned} \quad (20)$$

Next to the boundary, near-second-order accuracy can be maintained by the third- and fifth-order schemes by using the following

$$\phi_{j+1/2} = \epsilon \left[\Delta E_{j+1/2}^+ - \Delta E_{j+1/2}^- \right] \quad (21)$$

For $\epsilon = 0$, this flux leads to a second-order central difference. For $\epsilon = 1$, this is the same as the first-order dissipation term given by equation (15). By including a nonzero ϵ , dissipation is added to the second-order, central-difference scheme to help suppress any oscillations. A value of $\epsilon = 0.01$ is used for all of the results presented in this paper.

IMPLICIT SCHEME

This section describes the way in which equation (3) is numerically represented and solved. Application of a first-order backward Euler formula to this system of equations yields the following delta-form equation

$$\left[\frac{1}{J\Delta\tau} I + \left(\frac{\partial R}{\partial D} \right)^n \right] (D^{n+1} - D^n) = -R^n \quad (22)$$

where the superscript n is the pseudotime iteration count and the vector R is the residual vector. When the formula given in equation (13) is followed and a second-order, central-difference formula is applied to the viscous terms, at a point $x_{j,k}, y_{j,k}$ the numerical approximation to the residual vector is given by

$$R_{j,k} = \frac{\tilde{E}_{j+1/2,k} - \tilde{E}_{j-1/2,k}}{\Delta\xi} + \frac{\tilde{F}_{j,k+1/2} - \tilde{F}_{j,k-1/2}}{\Delta\eta} - \frac{(\hat{E}_v)_{j+1,k} - (\hat{E}_v)_{j-1,k}}{2\Delta\xi} - \frac{(\hat{F}_v)_{j,k+1} - (\hat{F}_v)_{j,k-1}}{2\Delta\eta} \quad (23)$$

where the numerical fluxes \tilde{E} and \tilde{F} are evaluated using equation (14) with either the first-, third-, or fifth-order dissipation term given in equations (15,19,20), respectively. The generalized coordinates are chosen so that $\Delta\xi$ and $\Delta\eta$ are equal to one.

The formation of the exact Jacobian matrix of the residual vector will be too expensive for practical consideration, particularly when higher-order upwind differencing is used, so the implicit side formulation will be limited to using the residual resulting from the first-order upwind differencing. By applying the first-order dissipation term in equation (15) to the convective terms, the residual is given by

$$R_{ij} = \frac{1}{2} \left[\hat{E}_{j+1,k} - \hat{E}_{j-1,k} - \Delta E_{j+1/2,k}^+ + \Delta E_{j+1/2,k}^- + \Delta E_{j-1/2,k}^+ - \Delta E_{j-1/2,k}^- \right. \\ \left. + \hat{F}_{j,k+1} - \hat{F}_{j,k-1} - \Delta F_{j,k+1/2}^+ + \Delta F_{j,k+1/2}^- + \Delta F_{j,k-1/2}^+ - \Delta F_{j,k-1/2}^- \right. \\ \left. - (\hat{E}_v)_{j+1,k} + (\hat{E}_v)_{j-1,k} - (\hat{F}_v)_{j,k+1} + (\hat{F}_v)_{j,k-1} \right] \quad (24)$$

The exact Jacobian matrix of the residual vector will form a banded matrix of the form

$$\frac{\partial R}{\partial D} = \mathcal{B} \left[\frac{\partial R_{j,k}}{\partial D_{j,k-1}}, 0, \dots, 0, \frac{\partial R_{j,k}}{\partial D_{j-1,k}}, \frac{\partial R_{j,k}}{\partial D_{j,k}}, \frac{\partial R_{j,k}}{\partial D_{j+1,k}}, 0, \dots, 0, \frac{\partial R_{j,k}}{\partial D_{j,k+1}} \right] \quad (25)$$

where \mathcal{B} refers to a banded matrix. By using approximate Jacobians of the flux differences as derived and analyzed by Barth [22], the implicit side of the numerical equation is formed using the following terms

$$\begin{aligned}
\frac{\partial R_{j,k}}{\partial D_{j,k-1}} &\approx \frac{1}{2}(-\hat{B}_{j,k-1} - B_{j,k-1/2}^+ + B_{j,k-1/2}^-) + (\gamma_2)_{j,k-1} \\
\frac{\partial R_{j,k}}{\partial D_{j-1,k}} &\approx \frac{1}{2}(-\hat{A}_{j-1,k} - A_{j-1/2,k}^+ + A_{j-1/2,k}^-) + (\gamma_1)_{j-1,k} \\
\frac{\partial R_{j,k}}{\partial D_{j,k}} &\approx \frac{1}{2}(A_{j+1/2,k}^+ + A_{j-1/2,k}^+ - A_{j+1/2,k}^- - A_{j-1/2,k}^- \\
&\quad B_{j,k+1/2}^+ + B_{j,k-1/2}^+ - B_{j,k+1/2}^- - B_{j,k-1/2}^-) \\
\frac{\partial R_{j,k}}{\partial D_{j+1,k}} &\approx \frac{1}{2}(\hat{A}_{j+1,k} - A_{j+1/2,k}^+ + A_{j+1/2,k}^-) - (\gamma_1)_{j+1,k} \\
\frac{\partial R_{j,k}}{\partial D_{j,k+1}} &\approx \frac{1}{2}(\hat{B}_{j,k+1} - B_{j,k+1/2}^+ + B_{j,k+1/2}^-) - (\gamma_2)_{j,k+1}
\end{aligned} \tag{26}$$

where $\hat{A} = \hat{A}_1$, $\hat{B} = \hat{A}_2$ in equation (8), and

$$\begin{aligned}
A^\pm &= X_1 \Lambda_1^\pm X_1^{-1} \\
B^\pm &= X_2 \Lambda_2^\pm X_2^{-1}
\end{aligned}$$

Only the orthogonal mesh terms are retained for the implicit viscous terms resulting in

$$\begin{aligned}
\gamma_1 &= \frac{\nu}{J}(\xi_x^2 + \xi_y^2) I_m \frac{\partial}{\partial \xi} \\
\gamma_2 &= \frac{\nu}{J}(\eta_x^2 + \eta_y^2) I_m \frac{\partial}{\partial \xi}
\end{aligned} \tag{27}$$

The matrix I_m is a modified identity matrix given by

$$I_m = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The numerical system of equations thus formed is solved using a line-relaxation method. In this procedure, the entire numerical matrix equation is first formed from values at the previous time level. At this point the numerical equation is stored as a banded matrix of the form

$$\mathcal{B}[V, 0, \dots, 0, X, Y, Z, 0, \dots, 0, W] \Delta D = \hat{R}$$

where $\Delta D = D^{n+1} - D^n$ and V, W, X, Y , and Z are vectors of 3 by 3 blocks which lie on the diagonals of the banded matrix, with the Y vector on the main diagonal. This matrix is approximately solved using an iterative approach. One family of lines is used as the sweep direction. Using, for example, the ξ family, a tridiagonal matrix is formed by multiplying the elements outside the

tridiagonal band by the current ΔD and shifting them over to the right-hand side. This can be represented by the following

$$\mathcal{B}[X, Y, Z](\Delta D)^{l+1} = \hat{R} - \Delta D_{j,k-1}^l V - \Delta D_{j,k+1}^l W$$

where l is an iteration index. This system can be solved most efficiently by first performing and storing the LU decomposition of the tridiagonal matrix before the iteration is begun. Then for each iteration, the right-hand side is formed using the latest known ΔD , and the entire system is back-solved. The LU decomposition can be entirely vectorized, but the backsolution is inherently recursive and cannot be vectorized.

UNSTEADY FORMULATION

The current scheme can be easily extended to solve unsteady problems with the use of subiterations in pseudotime at each physical time step. The details of this formulation are given by Rogers and Kwak [10] and by Athavale and Merkle [11], and a short summary of this follows. First, the time derivative in the momentum equations is discretized using a second-order backward Euler formula, resulting in

$$\frac{I_m}{\Delta t} (1.5 \hat{D}^{n+1} - 2 \hat{D}^n + 0.5 \hat{D}^{n-1}) = -R^{n+1}$$

where R is the same residual vector as in equation (23). Here physical time is denoted by t and the superscript n denotes the solution at time $t = n\Delta t$. This equation leaves no way to update the pressure to the next time level because of the I_m matrix on the left-hand side. Here the continuity equation is replaced with an artificial compressibility relation and a pseudotime level is introduced, resulting in

$$I_{t\tau} (\hat{D}^{n+1,m+1} - \hat{D}^{n+1,m}) = -R^{n+1,m+1} - \frac{I_m}{\Delta t} (1.5 \hat{D}^{n+1,m} - 2 \hat{D}^n + 0.5 \hat{D}^{n-1})$$

Here pseudotime is denoted by τ and the superscript m represents a subiteration index in pseudotime. The matrix $I_{t\tau}$ is a diagonal matrix given by

$$I_{t\tau} = \text{diag} \left[\frac{1}{\Delta \tau}, \frac{1.5}{\Delta t}, \frac{1.5}{\Delta t} \right]$$

After linearizing the residual about the $n+1, m$ time level, the following equation is obtained

$$\left[\frac{I_{t\tau}}{J} + \left(\frac{\partial R}{\partial D} \right)^{n+1,m} \right] (\hat{D}^{n+1,m+1} - \hat{D}^{n+1,m}) = -R^{n+1,m} - \frac{I_m}{\Delta t} (1.5 \hat{D}^{n+1,m} - 2 \hat{D}^n + 0.5 \hat{D}^{n-1}) \quad (28)$$

It can be seen that this equation is very similar to its steady-state counterpart. The additional right-hand side terms and the different diagonal matrix on the left-hand side are the only differences between the two. This makes it quite simple to program a code capable of using this scheme to solve both unsteady and steady-state problems.

BOUNDARY CONDITIONS

Implicit boundary conditions are used at all of the boundaries; this helps make possible the use of large time steps. At a no-slip surface, the velocity is specified to be zero, and the pressure at the boundary is obtained by specifying that the pressure gradient normal to the wall be zero. The boundary conditions used for inflow and outflow regions are based on the method of characteristics. The formulation of these boundary conditions is similar to that given by Merkle and Tsai [23], but the implementation is slightly different. The scheme is derived here for a $\xi = \text{constant}$ boundary, with similar results for a $\eta = \text{constant}$ boundary. The finite-speed waves which arise with the use of artificial compressibility are governed by the following

$$\frac{\partial \hat{D}}{\partial \tau} = -\frac{\partial \hat{E}}{\partial \xi} = -\frac{\partial \hat{E}}{\partial D} \frac{\partial D}{\partial \xi} = -\hat{A} \frac{\partial D}{\partial \xi} = -X \Lambda X^{-1} \frac{\partial D}{\partial \xi}$$

then

$$X^{-1} \frac{\partial \hat{D}}{\partial \tau} = -\Lambda X^{-1} \frac{\partial D}{\partial \xi} \quad (29)$$

If one were to move the X^{-1} matrix inside the spatial and time derivative, then it can be seen that this would be a system of scalar equations, each with the form of a wave equation. The sign of the eigenvalues in the Λ matrix determines the direction of travel of the wave. For each positive (negative) eigenvalue, there is a wave propagating information in the positive (negative) ξ direction. The number of positive or negative eigenvalues determines the number of characteristic waves propagating information from the interior of the computational domain to the boundary. Thus, at the boundary we will use these characteristics which bring information from the interior as part of our boundary conditions. The rest of the information should come from outside the computational domain, and we are free to specify some variables.

There will be either one or two characteristics traveling toward the boundary from the interior because there is always at least one positive eigenvalue and one negative eigenvalue. In order to select the proper characteristic waves, equation (29) is multiplied by a diagonal selection matrix L which has an entry of 1 in the position of the eigenvalue we wish to select, and zeros elsewhere. Thus

$$L X^{-1} \frac{\partial \hat{D}}{\partial \tau} = -L \Lambda X^{-1} \frac{\partial D}{\partial \xi} \quad (30)$$

Replacing the time derivative with an implicit Euler time step gives

$$\left(\frac{L X^{-1}}{J \Delta \tau} + L \Lambda X^{-1} \frac{\partial}{\partial \xi} \right) (D^{n+1} - D^n) = -L \Lambda X^{-1} \frac{\partial D^n}{\partial \xi} \quad (31)$$

This gives either one or two relations, depending on the number of nonzero elements in L . To complete the set of equations, some variables must be specified to be constant. Here is defined a vector Ω of the variables to be held constant, such that

$$\frac{\partial \Omega}{\partial \tau} = 0 \longrightarrow \frac{\partial \Omega}{\partial D} \frac{\partial D}{\partial \tau} = 0 \longrightarrow \frac{\partial \Omega}{\partial D} = 0 \quad (32)$$

Combining equations (31) and (32) gives

$$\left(\frac{L X^{-1}}{J \Delta \tau} + L \Lambda X^{-1} \frac{\partial}{\partial \xi} + \frac{\partial \Omega}{\partial D} \right) (D^{n+1} - D^n) = -L \Lambda X^{-1} \frac{\partial D^n}{\partial \xi} \quad (33)$$

Equation (33) can be used to update the variables implicitly at any of the inflow or outflow boundaries with the proper choice of L and Ω .

Inflow Boundary

At the inflow, there will be one characteristic wave traveling out of the computational domain since fluid is traveling into the domain. If the incoming fluid is traveling in the positive ξ direction, then

$$\begin{aligned} Q &> 0 \\ Q + c &> 0 \\ Q - c &< 0 \end{aligned}$$

This third eigenvalue will be the one we wish to select, and so L will have a 1 for the third diagonal entry. If the incoming fluid is traveling in the negative ξ direction, then

$$\begin{aligned} Q &< 0 \\ Q + c &> 0 \\ Q - c &< 0 \end{aligned}$$

and the second eigenvalue is the one corresponding to the wave propagation out of the computational domain, requiring a 1 in the second diagonal entry of L .

Two different sets of specified variables have been used successfully for inflow boundaries. One set consists of the total pressure and the cross-flow velocity. This set is useful for problems in which the inflow velocity profile is not known. For this set the Ω vector is

$$\Omega = \begin{bmatrix} p + \frac{1}{2}(u^2 + v^2) \\ 0 \\ v \end{bmatrix}; \quad \frac{\partial \Omega}{\partial D} = \begin{bmatrix} 1 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The second possible set of specified variables consists of the velocity components. This is useful for problems in which a specific velocity profile is desired at the inflow. The Ω vector for this is

$$\Omega = \begin{bmatrix} 0 \\ u \\ v \end{bmatrix}; \quad \frac{\partial \Omega}{\partial D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Outflow Boundary

At the outflow boundary there are two characteristic waves traveling out of the computational domain since fluid is also leaving the domain. If the fluid is traveling along the positive ξ direction, then

$$\begin{aligned} Q &> 0 \\ Q + c &> 0 \\ Q - c &< 0 \end{aligned}$$

and we require a 1 in the first two diagonal entries of the L matrix. If the fluid is traveling in the negative ξ direction, then

$$\begin{aligned} Q &< 0 \\ Q + c &> 0 \\ Q - c &< 0 \end{aligned}$$

and we require a 1 in both the first and third diagonal entries of the L matrix.

For all of the test problems presented in this paper, static pressure was specified at the outflow boundary, resulting in

$$\Omega = \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix}; \quad \frac{\partial \Omega}{\partial D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

COMPUTED RESULTS

The code was run for three 2-D laminar-flow test cases. These are a driven square-cavity flow, flow over a backward-facing step, and flow over a circular cylinder. These cases were chosen because they each have been studied previously by others in experiments or in computations. The computing times reported here are the CPU seconds used on a Cray 2. For comparison, these times are nearly the same as those obtained running on a Cray XMP-48. The computations are run until the maximum residual has converged over six orders of magnitude, the maximum divergence of velocity over all the points is less than 10^{-4} , and the flow quantities being measured have approached a steady-state value in at least four significant digits.

For each of the test cases presented, the larger the time step $\Delta \tau$, the better the convergence was, provided the solution remained stable. In all of the cases presented here the solution remained stable no matter how large a time step was used, so the time step was set to 10^{12} , which effectively reduced the $1/\Delta \tau$ term to zero. The choice of β for each case was arrived at through numerical convergence tests. It was found that the convergence was quite sensitive to the value of β , and in some cases, the choice of β could cause the scheme to become unstable. For most cases, however, it was easy to find a range of β for which the code would converge very quickly. The convergence of the current formulation is degraded by the errors introduced by the approximate Jacobians on the left-hand side of the equations and by the fact that the whole system of equations is not exactly solved by the line-relaxation process. If it were possible to use the exact Jacobians and solve the system exactly, then this would be a Newton iteration, in which case one would expect to have quadratic convergence when using a very large time step for any value of β . Analysis of these errors and their relationship to β is under way, and it is hoped that a guideline for choosing β and for minimizing the eigenvalues of the amplification matrix can be obtained. Until such a guideline is found, the numerical tests will have to suffice.

Driven Cavity Flow

The 2-D flow in a driven square cavity whose top wall moves with a uniform velocity has been used rather extensively as a validation test case by several authors in the recent past. It provides a good test case in that there is no primary flow direction and the boundary conditions are very

simple to use. Ghia, Ghia, and Shin [24] presented extensive numerical data obtained from their multigrid vorticity-stream function formulation using very fine grids. They reported results which agreed well with other computational efforts. Other recent computational work involving this particular geometry include Schreiber and Keller [25] who use a vorticity-stream function formulation; Kim and Moin [6] who use a fractional-step method in primitive variables in conjunction with approximate factorization; Vanka [26] who uses a multigrid technique in primitive variables; and Benjamin and Denny [27] who use a centrally differenced vorticity-stream function formulation in conjunction with an ADI scheme.

The current calculations attempt to maintain the accuracy of these authors while using fewer grid points. The flow is calculated for Reynolds numbers of 100, 400, 1000, 3200, 5000, 7500, and 10,000 using a grid of 81 by 81 points where the points are clustered toward the walls. This grid is shown in figure 1. The value of the artificial compressibility β was set to 20 for the Reynolds number of 100, to 10 for the 400 Reynolds number case, to 2 for a Reynolds number of 1000, and to 1 for the higher Reynolds numbers. The implicit line relaxation used 11 sweeps in the ξ -direction for each iteration.

The velocity components on the lines passing through the geometric center of the cavity are compared to the results of Ghia, Ghia, and Shin [24] in figure 2. The u-velocity component is plotted along the y-axis for the different Reynolds numbers in figure 2a. The origins of the plots have been shifted to the left for each successive Reynolds number case. The data of Ghia were obtained from a uniform grid of 129 by 129 points for Reynolds numbers up to 3200, and a uniform grid of 257 by 257 points for Reynolds numbers 5000 and greater. It is noted that these two computed results agree very well with each other. In figure 2b, the v-velocity component is plotted along the x-axis passing through the geometric center for the different Reynolds numbers. The origins of these plots are shifted up for each successive Reynolds number case. Again, good agreement is seen between the two computed results.

In table 1, the stream function and vorticity quantities are given for the core of the primary vortex for all the Reynolds numbers. Included with the present results are the results of Ghia, Ghia, and Shin [24], Schreiber and Keller [25], and Kim and Moin [6]. Listed below the flow quantities is the grid size used for the calculation. Good agreement among all calculations is seen in the lower Reynolds number cases. The discrepancies between different solutions increase at the higher Reynolds numbers, although the same general trend of a leveling off and then a slight decrease in the value of the stream-function is seen.

To study in more detail the 10,000 Reynolds number case, the streamlines are plotted in figure 3. The values of the stream-function contours for this plot are given in table 2. The contour levels plotted correspond with the values plotted by Ghia, Ghia, and Shin [24] for this case. Qualitatively, the plots appear to be identical. They each show secondary vortices of the same size and shape in the lower corners and the upper left corner. In table 3, the stream-function, vorticity, and location of the vortex core for all the secondary vortices for this 10,000 Reynolds number case are given for both the present results and the results of Ghia, Ghia, and Shin [24]. In this table, the initial T stands for top, B stands for bottom, R stands for right, L stands for left, and the superscript number corresponds to the level of the secondary vortex. Thus BR³ refers to the third and smallest secondary vortex found in the bottom right corner of the cavity. Good agreement between the two computations is seen for this case, especially considering that the results of Ghia, Ghia, and Shin [24] uses over 10 times as many grid points (66,049 versus 6561).

The convergence toward a steady-state for this problem was very good for the three lowest Reynolds number cases, which required less than 100 iterations and only 35 sec of computing time. The higher Reynolds number cases were slower to converge; the 10,000 case took 550 iterations

and 215 sec of computing time. The average of the computing requirements for all seven cases came out to 250 iterations and 100 sec of computing time.

Flow Over a Backward Facing Step

A second 2-D problem which has been used as a validation case is the flow over a backward-facing step. The challenge in modeling this problem comes from the fact that the size of the separation bubbles downstream of the step are very sensitive to the pressure gradient in the flow. The geometry used in the calculations is shown in figure 4. At the inflow boundary, a parabolic profile is prescribed throughout the calculation, and the static pressure is allowed to change. Two step heights downstream from the inflow a two-to-one expansion is encountered. The outflow boundary extends to 30 step heights downstream of the step. The ability of the flow code to predict the reattachment length, x_1 , of the primary separation bubble behind the step, as well as the separation and reattachment locations, x_2 and x_3 , of the secondary separation bubble on the opposite wall, was tested by comparing the computed results to experimental values given by Armaly *et al.* [28]. These quantities were measured for the laminar range of Reynolds numbers, which are based on the average inflow velocity and twice the step height. The flow was calculated using a grid of 100 points in the streamwise direction and 53 points in the crossflow direction. The initial conditions were specified to be free-stream velocity at the interior points with uniform pressure everywhere. For the Reynolds numbers of 100 and 200, β was set to 1, for the Reynolds number of 300 case, 0.5 was used, and for the Reynolds numbers of 400 through 800, β was set to 0.1. The implicit line-relaxation process used sweeps along the primary flow direction.

In figure 5, the quantities x_1 , x_2 , and x_3 are plotted versus Reynolds number for both the present computed results and the experimental results of Armaly *et al.* [28]. Good agreement is seen between the two for the value of x_1 at the lower Reynolds numbers before the secondary separation appears. At a Reynolds number of 400, the secondary separation bubble is present, and the computed primary reattachment length begins to fall off of the experimental curve. Similarly, the computed secondary separation points are shorter than the experimental values, although the same behavior is seen; that is that the secondary separation point is upstream of the primary reattachment point. The computed secondary reattachment point is seen to be close to the experimental values. In their experiment, Armaly *et al.* reported that the flow was found to be 3-D near the step for Reynolds numbers greater than 400, and that the 3-D effects were negligible for lower Reynolds numbers. These 3-D effects could explain the discrepancies between the calculations and experiment.

Results similar to the present results were reported by Kim and Moin [6]. They reported a primary reattachment length of just under 12 step heights for a Reynolds number of 800, and the present result for this Reynolds number is 11.48. They reported a secondary separation bubble size of 7.8 and 11.5 step heights for Reynolds numbers of 600 and 800, respectively. The present results show secondary separation bubble sizes of 7.34 and 11.07 step heights for these two Reynolds numbers. The similarities between these computational results give credence to the idea that the three-dimensionality of the flow affects the separation bubble size.

The convergence characteristics of the code for this problem are very good. In figure 6 the convergence histories of the $Re = 100$ and 800 cases are plotted. Figure 6a plots the log of the maximum residual normalized by the maximum residual at the first time step versus iteration number for the $Re = 100$ and 800 cases. Figure 6b plots the primary reattachment length x_1 versus iteration number. The $Re = 100$ case converges within 55 iterations and the $Re = 800$ case converges within 165 iterations. The average number of iterations required for the eight different Reynolds number cases is 104 and the average required computing time is just under 11.5 sec.

Flow Over a Circular Cylinder

As an example of an external flow problem, the flow over a 2-D circular cylinder was calculated. The grid was an algebraically generated o-grid with 100 points in the circumferential direction and 60 points radially. The grid points were clustered radially toward the body and the outer boundary was placed 10 diameters from the cylinder. The code was run and steady-state solutions were obtained for the Reynolds numbers of 5, 10, 20, and 40, based on the free-stream velocity and the cylinder diameter. The value of the artificial compressibility constant β was set to 50 for all the cases. At the outer boundary, where fluid was entering the domain, the velocity was held constant, and where fluid was leaving the domain, the static pressure was held constant. The line-relaxation scheme used four sweeps in both of the coordinate directions, which seemed to be the best trade-off of convergence versus computing time in numerical tests.

For each case, several flow quantities of the flow were computed. Figure 7 shows a schematic diagram of the geometry for this flow problem along with several of these flow quantities. These quantities are the flow separation length measured from the rear of the cylinder in cylinder diameters (L_{sep}), the angle which defines the point of separation from the body (θ_{sep}), the coefficient of drag (C_D), the coefficient of pressure drag (C_{Dp}), and the coefficient of pressure at the front (C_{pf}) and rear (C_{pr}) stagnation points. In table 4, these quantities are presented for the present calculations as are the numerical results of Takami and Keller [29], Dennis and Chang [30], Tuann and Olson [31], Braza, Chassaing, and Minh [32], and the experimental results of Coutanceau and Bouard [33] and of Tritton [34]. The comparisons show that there is very good agreement among nearly all of the results.

The convergence of the code toward a steady-state solution for the problem of a circular cylinder was found to be good. All four Reynolds number cases converged in less than 70 iterations, requiring an average of 21 sec of computing time.

CONCLUSIONS

The use of a flux-difference split upwind-differencing scheme has been applied to the artificial compressibility equations to solve the steady-state Navier-Stokes equations. This eliminates the need for any explicitly added artificial dissipation terms and any artificial dissipation coefficients. Although no direct comparison has been made, this code has been found to be much more robust and easier to run than previous applications by the authors of the artificial compressibility method which used central differencing plus artificial dissipation. The natural addition of dissipation through the use of upwind-biased stencils requires no trial-and-error adjusting of smoothing parameters as does the artificial dissipation. As well, it is thought that the terms on the main diagonal of the implicit side matrix which are not present in the central difference scheme make the scheme much more robust. Implicit boundary conditions based on the method of characteristics were presented. The accuracy of the upwind scheme has been established using three 2-D test cases. Good comparison was found between the current method and other methods which used many more grid points in calculating the flow inside a driven cavity. Results which compared well with experimental values were obtained in calculating the flow over a backward-facing step. The discrepancy at higher Reynolds numbers could be explained by the three-dimensionality of the experiment, and this was supported by other computational results. Finally, good comparison was found in measuring the flow around a circular cylinder. Perhaps the most striking feature of the current code is its ability to obtain steady-state solutions in a small number of iterations for most problems. Very good convergence rates were observed when the proper choice of the artificial compressibility constant β was made. No analytical guidelines for the choice of this parameter have been derived as yet, and

this is a subject of ongoing work. The extension of the current code to three dimensions is nearly completed and has been found to be straightforward.

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TABLE 1
 STREAM-FUNCTION AND VORTICITY AT THE CENTER OF THE
 PRIMARY VORTEX FOR DIFFERENT REYNOLDS NUMBERS

Re	Present $\psi_{min} (\omega_{v.c.})$	Ghia <i>et al.</i> [24] $\psi_{min} (\omega_{v.c.})$	Schreiber <i>et al.</i> [25] $\psi_{min} (\omega_{v.c.})$	Kim <i>et al.</i> [6] $\psi_{min} (\omega_{v.c.})$
100	-0.1030(-3.104) 81x81	-0.1034(-3.166) 129x129	-0.1033(-3.182) 121x121	-0.1030(-3.177) 65x65
400	-0.1131(-2.296) 81x81	-0.1139(-2.294) 129x129	-0.1130(-2.281) 141x141	-0.1120(-2.260) 65x65
1000	-0.1171(-2.044) 81x81	-0.1179(-2.050) 129x129	-0.1160(-2.026) 141x141	-0.1160(-2.026) 97x97
3200	-0.1195(-1.904) 81x81	-0.1204(-1.989) 129x129	—	-0.1150(-1.901) 97x97
5000	-0.1192(-1.846) 81x81	-0.1190(-1.860) 257x257	—	-0.1120(-1.812) 97x97
7500	-0.1186(-1.846) 81x81	-0.1200(-1.880) 257x257	—	—
10000	-0.1177(-1.826) 81x81	-0.1197(-1.881) 257x257	-0.1028(-1.622) 180x180	—

TABLE 2
 VALUES FOR STREAMLINE CONTOURS IN FIGURE 3

Contour number	Value of ψ	Contour letter	Value of ψ
1	1.0×10^{-5}	A	-1.0×10^{-5}
2	5.0×10^{-5}	B	-1.0×10^{-4}
3	1.0×10^{-4}	C	-0.01
4	2.5×10^{-4}	D	-0.03
5	5.0×10^{-4}	E	-0.05
6	1.0×10^{-3}	F	-0.07
7	1.5×10^{-3}	G	-0.09
8	3.0×10^{-3}	H	-0.1
		I	-0.11
		J	-0.115

TABLE 3
 PROPERTIES OF THE SECONDARY VORTICES FOR THE DRIVEN CAVITY AT $Re = 10,000$

Vortex	Results	$\psi_{v.c.}$	$\omega_{v.c.}$	$x_{v.c.}$	$y_{v.c.}$
TL	Present	2.418×10^{-3}	2.191	0.0723	0.9117
	Ghia, Ghia, and Shin [24]	2.420×10^{-3}	2.183	0.0703	0.9141
BL		1.434×10^{-3}	2.097	0.0585	0.1686
		1.518×10^{-3}	2.086	0.0586	0.1641
BR		3.227×10^{-3}	4.163	0.7619	0.0585
		3.418×10^{-3}	4.053	0.7656	0.0586
BL ²		-5.120×10^{-7}	-0.02207	0.1416	0.01722
		-7.757×10^{-7}	-0.02754	0.1560	0.01950
BR ²		-2.103×10^{-4}	0.3726	0.9277	0.07288
		-1.313×10^{-4}	0.3126	0.9336	0.06250
BR ³		-4.267×10^{-7}	-2.956×10^{-3}	0.9981	0.008697
		-5.668×10^{-9}	—	0.9961	0.003900

TABLE 4
FLOW QUANTITIES FOR A CIRCULAR CYLINDER

Source	Reynolds Number			
	5	10	20	40
			L_{sep}	
Present	0	0.254	0.932	2.29
[29]	0	0.249	0.935	2.32
[30]	0	0.252	0.94	2.35
[31]	0	0.25	0.9	2.1
[33] (exp)	–	0.34	0.93	2.13
			θ_{sep} (degrees)	
Present	0	28.8	43.1	53.0
[29]	–	29.3	43.7	53.6
[30]	0	29.6	43.7	53.8
[31]	> 6	29.7	44.1	54.8
[32]	–	–	43.6	54.5
[33] (exp)	–	32.5	44.8	53.5
			C_D (C_{Dp})	
Present	4.18 (2.19)	2.89 (1.602)	2.08 (1.242)	1.549 (1.011)
[29]	–	2.80	2.01	1.536
[30]	4.12 (2.20)	2.85 (1.600)	2.05 (1.233)	1.522 (0.998)
[31]	4.66 (2.48)	3.18 (1.775)	2.25 (1.35)	1.675 (1.095)
[32]	–	–	2.18	1.60
[34] (exp)	4.16	3.06	2.02	1.65
			C_{pf} ($-C_{pr}$)	
Present	1.847 (1.067)	1.476 (0.755)	1.265 (0.615)	1.147 (0.536)
[29]	–	1.474 (0.670)	1.261 (0.537)	1.141 (0.512)
[30]	1.872 (1.044)	1.489 (0.742)	1.269 (0.589)	1.144 (0.509)
[31]	2.23 (1.081)	1.744 (0.773)	1.457 (0.614)	1.312 (0.543)

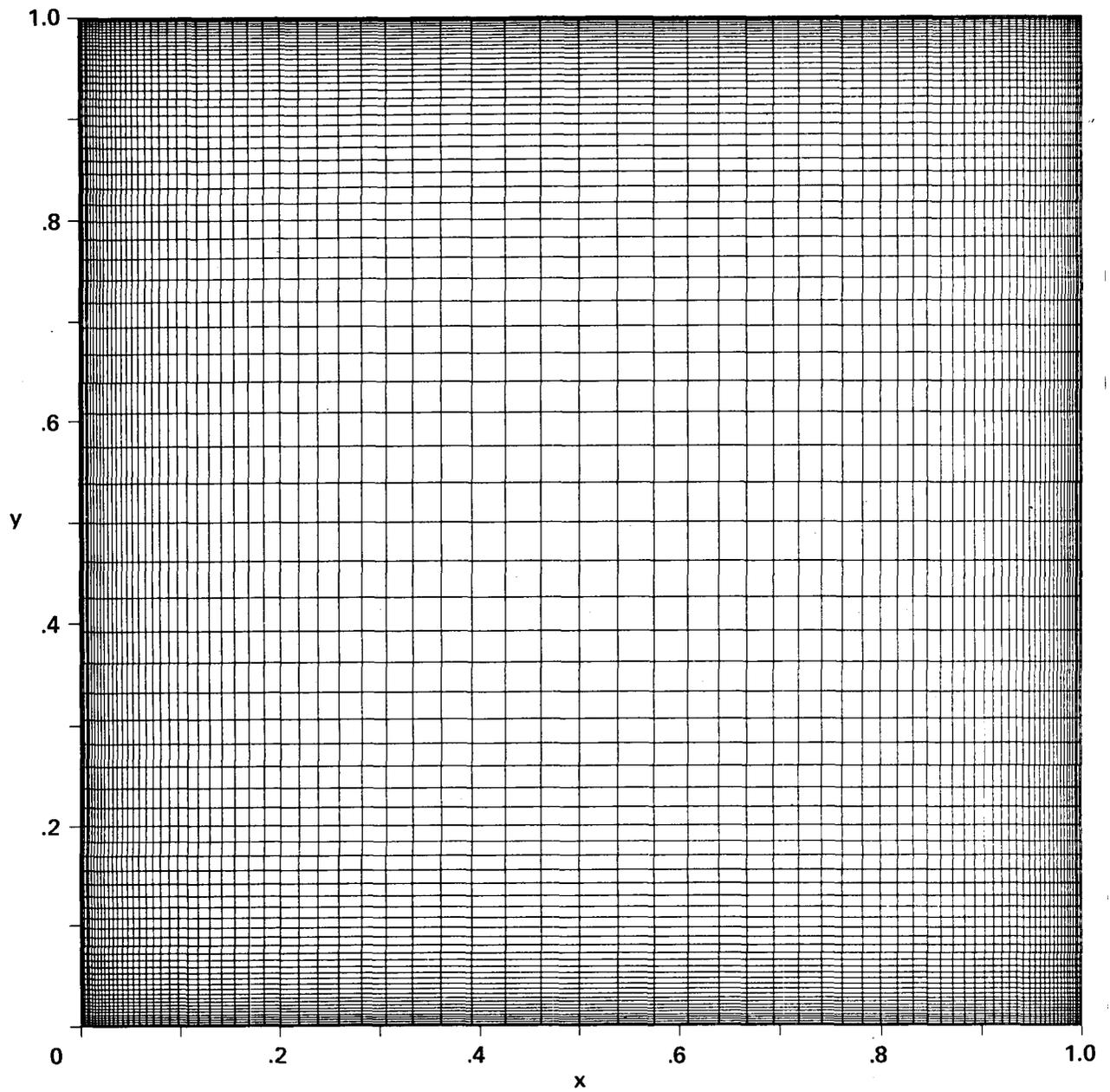


Fig. 1. Grid with 81x81 points used for computing the driven cavity flow.

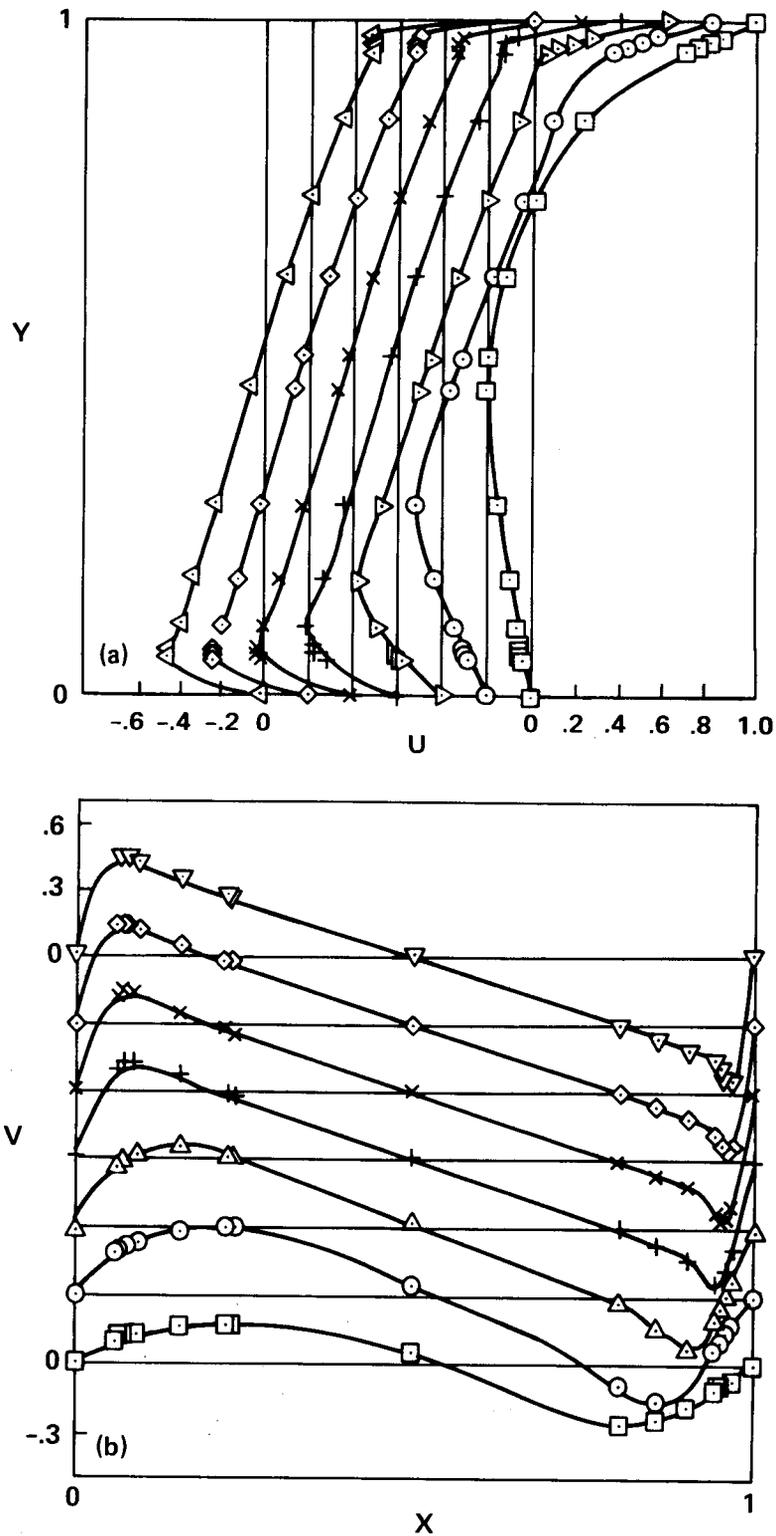


Fig 2. Comparison between present results (solid line) and computations of Ghia *et al.* [17] (symbols). \square : Re=100, \circ : Re=400, \triangle : Re=1000, $+$: Re=3200, \times : Re=5000, \diamond : Re=7500, and ∇ : Re=10,000. (a) U-velocity component versus y on the vertical centerline. (b) V-velocity component versus x on the horizontal centerline.

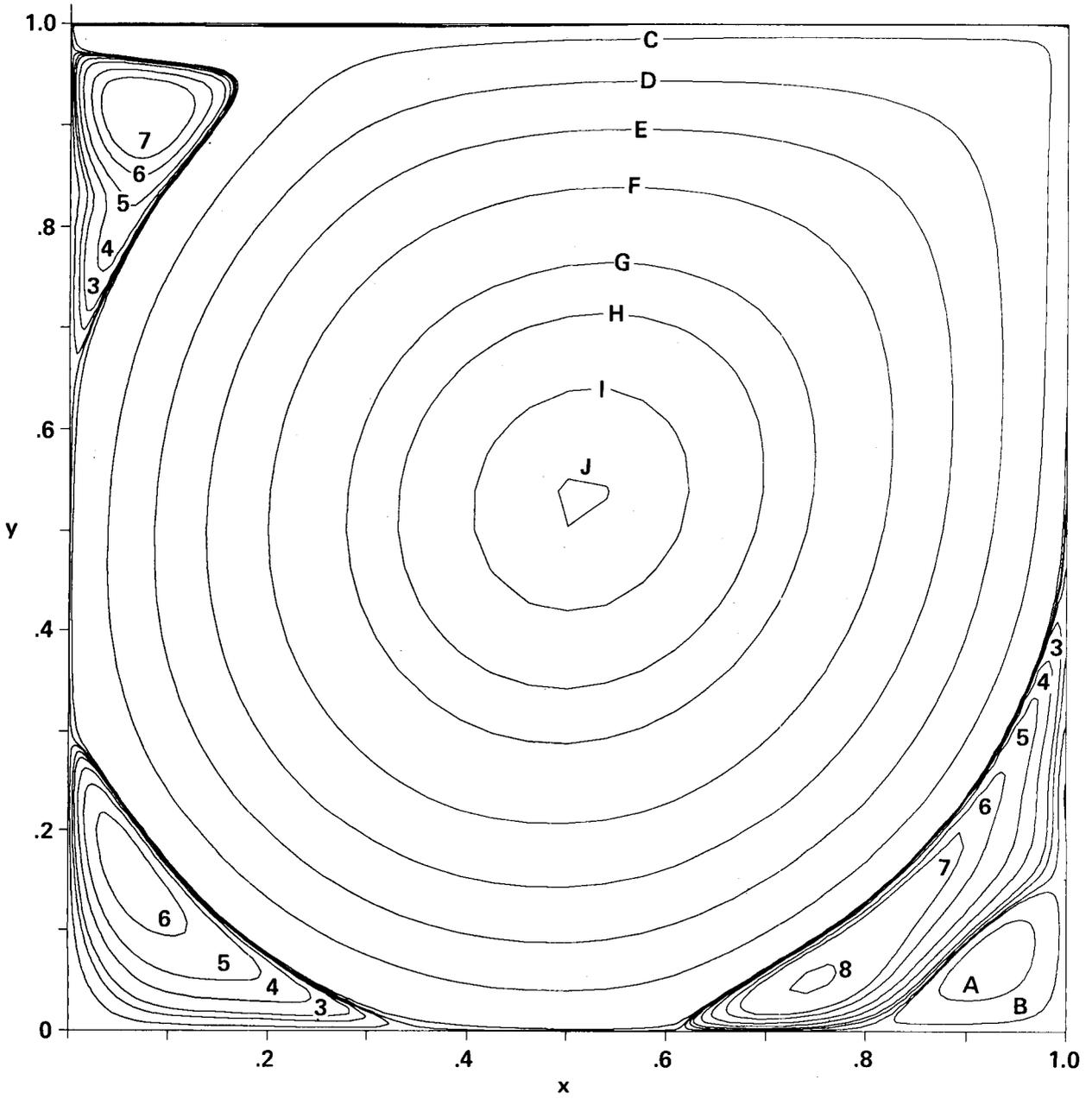


Fig. 3. Streamlines showing the driven cavity flow at $Re = 10,000$.

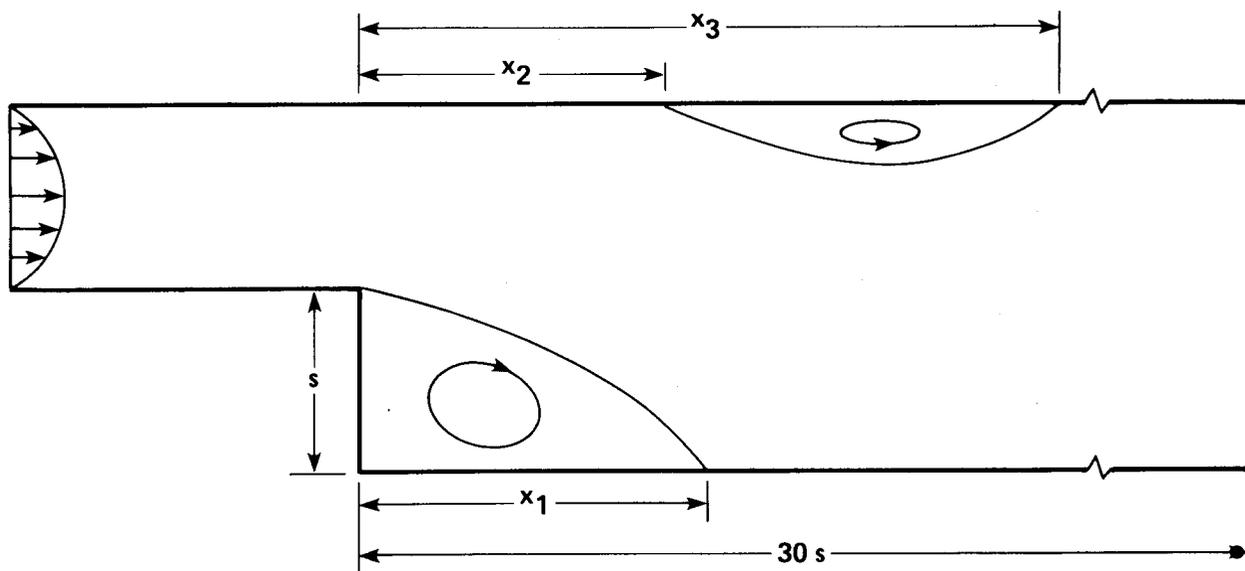


Fig. 4. Geometry of backward-facing step flow problem.

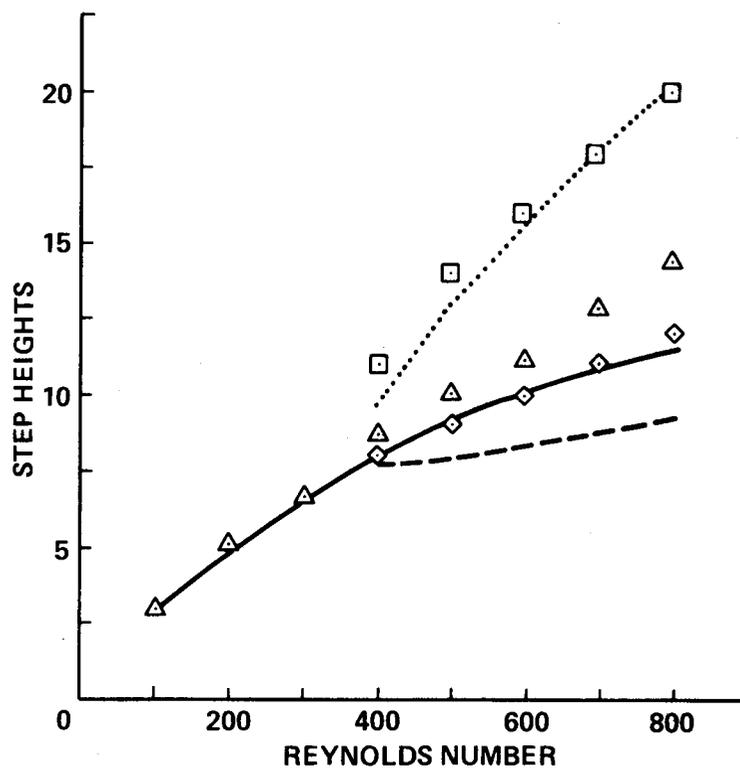


Fig. 5. Separation length versus Reynolds number for the flow over a backward-facing step. Solid line: computed x_1 , dashed line: computed x_2 , dotted line: computed x_3 , Δ : experimental x_1 , $+$: experimental x_2 , and \times : experimental x_3 .

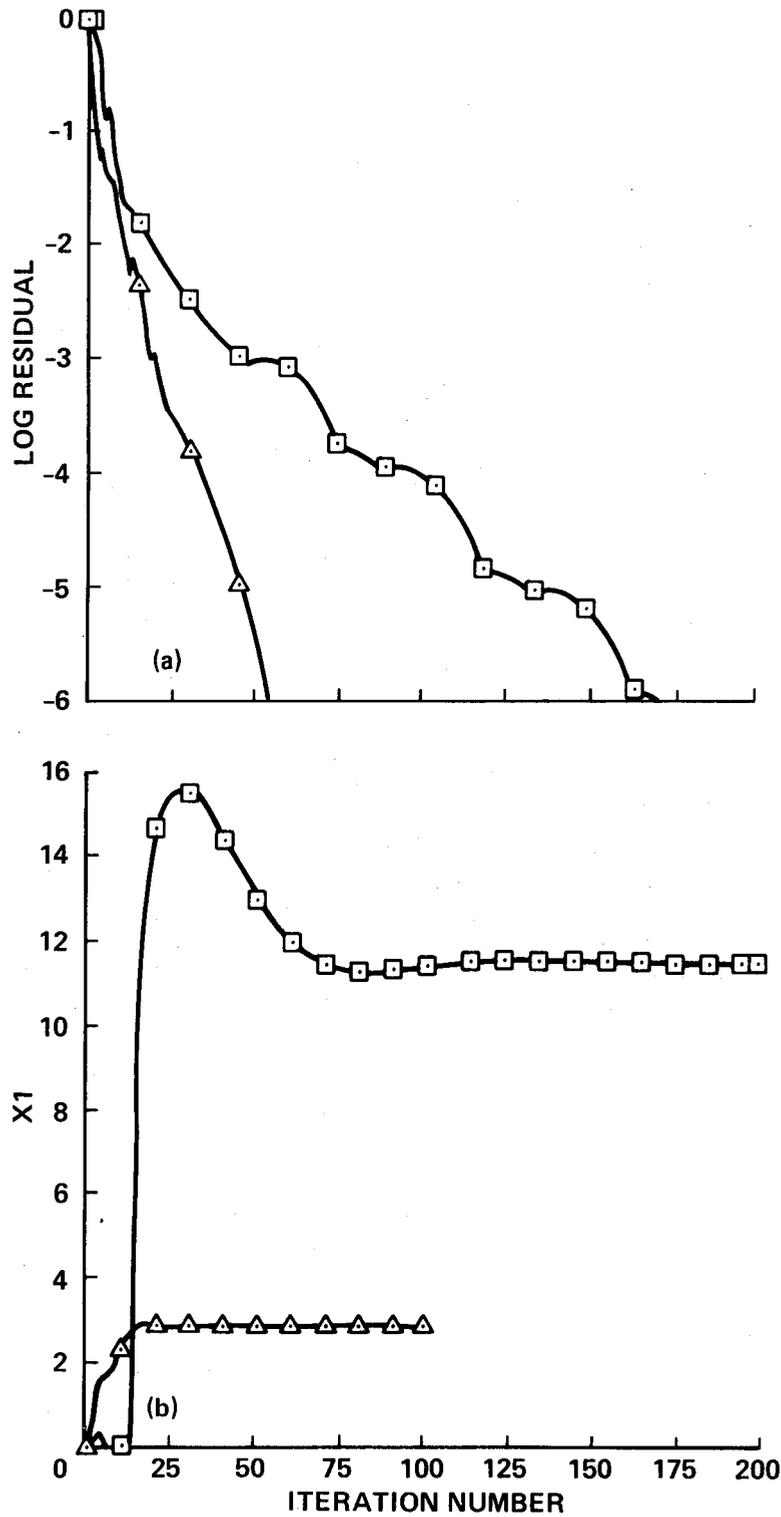


Fig. 6. Convergence history for the flow over a backward-facing step. \triangle : $Re = 100$, \times : $Re = 800$. (a) Log of the maximum residual versus iteration number. (b) Primary reattachment length versus iteration number.

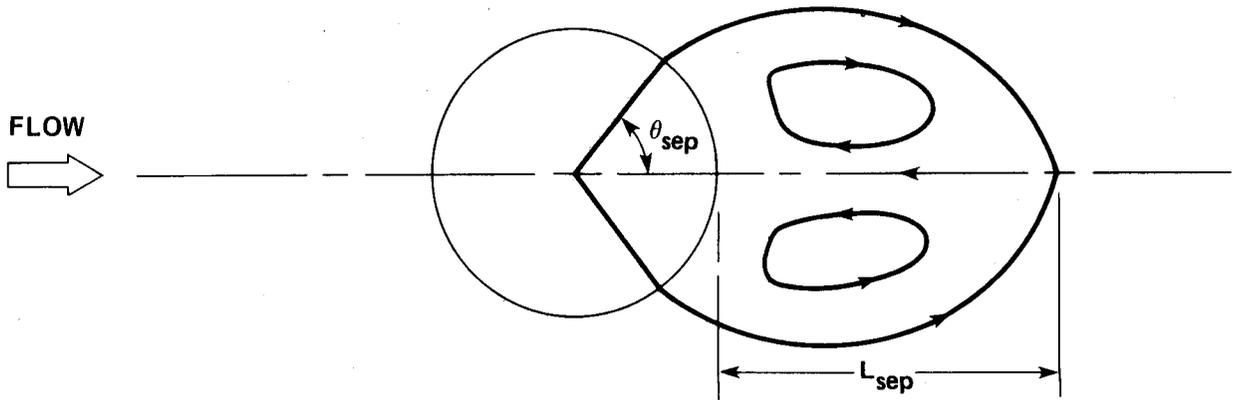


Fig. 7. Schematic diagram showing flow quantities for the circular cylinder flow computations.



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16. Abstract <p>The steady-state incompressible Navier-Stokes equations in two dimensions are solved numerically using the artificial compressibility formulation. The convective terms are upwind-differenced using a flux-difference split approach that has uniformly high accuracy throughout the interior grid points. The viscous fluxes are differenced using second-order-accurate central differences. The numerical system of equations is solved using an implicit line-relaxation scheme. Although the current study is limited to steady-state problems, it is shown that this entire formulation can be used for solving unsteady problems. Characteristic boundary conditions are formulated and used in the solution procedure. The overall scheme is capable of being run at extremely large pseudotime steps, leading to fast convergence. Three test cases are presented to demonstrate the accuracy and robustness of the code. These are the flow in a square-driven cavity, flow over a backward-facing step, and flow around a two-dimensional circular cylinder.</p>					
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